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ORIGINAL PAPER

Multilevel augmentation method with wavelet bases for singularly perturbed problem

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Abstract Multilevel augmentation method with wavelet bases is demonstrated to show as the fast technique for solving singularly perturbed problems. Linear and quadratic wavelet bases are employed for constructing the full form of matrix system. To reduce the size of matrix coefficients, the multilevel augmented technique is applied at each current basis level. It is found that the multilevel augmentation method is faster than the standard multilevel method at the same order of accuracy. Convergent rates for linear and quadratic bases are 2 and 4 respectively. By the application of wavelet bases, numerical accuracy can be easily improved by increasing just desired levels in the multilevel augmentation process.

Keywords Multilevel augmentation method · Singularly perturbed problem · Wavelet basis functions

1 Introduction

Singularly perturbed problem arises in various branches of applied science, for example, in the theory of fluid dynamics, chemical reactor, etc. It involves a small parameter multiplying the highest derivative in the governing equation. When the value of this parameter becomes very small, the unknown solution behaves very large change in a particular portion of domain. Standard numerical method usually fails to detect solution behavior [9]. In this work, the multilevel augmentation method is proposed to solve singularly perturbed problem. The main concept is using an approximate basis function for the solution space of the governing equation, and then projecting the terms

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of approximate solution on the functional basis space. This process provides residual that should be minimized with respect to the functional basis. By this concept, the accuracy of numerical solution depends directly on the type of basis function.

Wavelet basis functions, in the forms of linear and quadratic polynomials, are employed in the multilevel augmentation method. Full forms of stiffness matrices are presented. Wavelets in our consideration are compactly supported. It was presented previously by Chen et al. [5]. They introduced the multilevel augmentation method for solving certain boundary value problems. This method has also been applied to solve the sine-Gordon equation in [3] and some types of nonlinear boundary value problems in [2]. For solving the partial differential equations, the wavelet applications have been introduced by several authors, such as a wavelet-Galerkin method for solving parabolic equations [8], the singularly perturbed convection-dominated diffusion equation [6], non-homogeneous heat and wave equations [7], some types of elliptic problems [1], diffusion equation [4], and by a non-standard algorithm with a variable step size in [10].

In the present work, we consider both linear and quadratic wavelet bases in the multilevel augmentation method. Its advantages are fast and accurate for solving differential equations. We show that numerical accuracy is easily improved by increasing just wavelet levels in the multilevel augmentation method.

The paper is organized as follows. We introduce the concept of wavelet basis functions in Sect. 2. Details of the multilevel augmentation method using wavelet bases to solve numerically the singularly perturbed problem are presented in Sect. 3. Numerical results are shown in Sect. 4. Finally, we have made some conclusions in Sect. 5.

2 Construction of wavelet basis

The construction of the wavelet basis functions used in the Galerkin method follows the derivations proposed by Chen et al. [5]. They constructed multi-scale orthonormal bases for the Sobolev space $H_0^1(I)$ on the unit interval $I := [0, 1]$. Let X_n be the subspace of $H_0^1(I)$ whose elements are the piecewise polynomials of order k with knots j/μ^n, $j-1 \in Z_{\mu^n-1}$, where $Z_m := \{0, 1, 2, \ldots, m-1\}$, $\quad k > 2$ and $\mu > 1$ be a fixed positive integer. They have that

$$\mathrm{X}_0 = \mathrm{span}\left\{x^{j+1}(1-x) : j \in Z_{k-2}\right\},$$

and let W_n be the orthonormal complement of X_{n+1}in X_n, i.e.,

$$\mathrm{X}_n = \mathrm{X}_{n-1} \oplus^{\perp} \mathrm{W}_n,$$

and thus using this decomposition leads to

$$\mathrm{X}_n = \mathrm{X}_0 \oplus^{\perp} \mathrm{W}_1 \oplus^{\perp} \cdots \oplus^{\perp} \mathrm{W}_n.$$

After W_1 has been given, the spaces$\mathrm{W}_n$can be recursively constructed using the family of affine mappings $\Phi_\mu := \{\phi_e : e \in Z_\mu\}$ where

$$\phi_e(x) = \frac{x+e}{\mu}; \quad e \in Z_\mu.$$

Hence, the wavelet basis function $w_{ij} \in \mathrm{W}_i, i = 2, 3, \ldots, \quad j = 0, 1, \ldots, \dim \mathrm{W}_i - 1, \quad n = i - 1, l \in Z_r, r = \dim \mathrm{W}_1$ can be constructed by the following composition,

$$w_{ij} = \mu^{n\left(-\frac{1}{2}\right)} w_{1l} \circ \phi_e^{-1}(x); \; e \in Z_\mu^{i-1}. \tag{1}$$

where $Z_\mu^k = Z_\mu \times Z_\mu \times \cdots \times Z_\mu$, k times.

This construction will be applied to obtain both linear and quadratic wavelet bases used in our work in the next sections.

2.1 Linear wavelet basis

When choosing $k = 2, \mu = 2, r = 1$ and $\dim \mathrm{W}_i = 2^{i-1}$ for $i > \quad 0, l, Z_\mu$, and e are given by

$$l \in Z_r = \{0\}, \quad e \in Z_\mu = \{0, 1\},$$

and

$$\phi_0(x) = \frac{x}{2}, \quad \phi_1(x) = \frac{x+1}{2},$$

The desired basis of W_1 (level 1) is obtained by

$$w_{10}(x) = \begin{cases} x & ; x \in \left[0, \frac{1}{2}\right) \\ 1 - x & ; x \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{2}$$

The next basis level w_{2j} can be obtained from the mapping $w_{ij} = \mu^{\left(-\frac{1}{2}\right)} w_{1l} \circ \phi_e^{-1}(x)$ as shown in by the following.

Construction of W_2 from W_1

e	w_{1l}	$\Phi_e^{-1}(x)$	w_{2j}
0	w_{10}	$2x$	$w_{20}(x) = \begin{cases} \frac{1}{\sqrt{2}}(2x) & ; x \in \left[0, \frac{1}{4}\right) \\ \frac{1}{\sqrt{2}}(1-2x) & ; x \in \left[\frac{1}{4}, \frac{1}{2}\right] \end{cases}$
1	w_{10}	$2x - 1$	$w_{21}(x) = \begin{cases} \frac{1}{\sqrt{2}}(2x-1) & ; x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \frac{1}{\sqrt{2}}(2-2x) & ; x \in \left[\frac{3}{4}, 1\right] \end{cases}$

Next, w_{3j} can be obtained from $w_{3j} = \mu^{2\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_e^{-1}(x) = \frac{1}{2} w_{1l} \circ \phi_e^{-1}(x)$

Graphs of these three linear wavelet basis W_1, W_2, and W_3 are shown in Fig. 1. Any further linear wavelet levels can be obtained recursively by the same constructions.

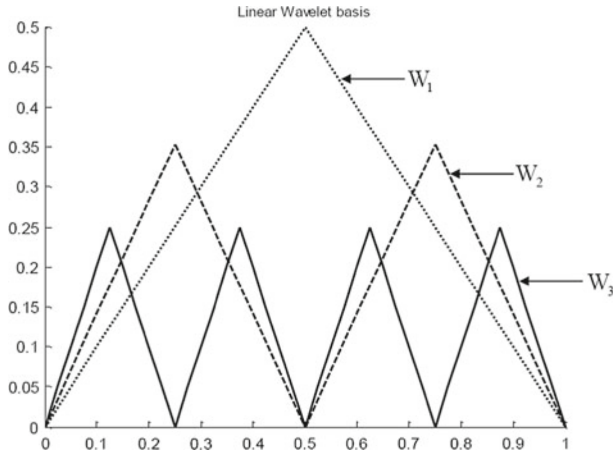

Fig. 1 Graphs of linear wavelet basis W_1, W_2 and W_3

2.2 Quadratic wavelet basis

In this case, we have to choose $k = 3$, $\mu = 2$, $r = 2$ and $\dim \mathrm{W}_i = 2^i$ for $i > 0$.

The desired quadratic bases W_0 and W_1 can be constructed by giving the mother level as

$$w_{00}(x) = \sqrt{3}x(1-x) \quad ; x \in [0, 1], \tag{3}$$

$$w_{10}(x) = \begin{cases} x(1-3x) & ; x \in \left[0, \frac{1}{2}\right) \\ (1-x)(3x-2) & ; x \in \left[\frac{1}{2}, 1\right] \end{cases}, \tag{4}$$

$$w_{11}(x) = \begin{cases} \sqrt{3}x(1-2x) & ; x \in \left[0, \frac{1}{2}\right) \\ \sqrt{3}(1-x)(1-2x) & ; x \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{5}$$

The second wavelet level W_2 is obtained by the mapping,

$$w_{2j} = \mu^{\left(-\frac{1}{2}\right)} w_{1l} \circ \phi_e^{-1}(x) = \frac{1}{\sqrt{2}} w_{1l} \circ \phi_e^{-1}(x).$$

Details of constructions are shown by the following.

Construction of W_2 from W_1

Similarly, the third level of quadratic basis W_3 is obtained by

$$w_{3j} = \mu^{2\left(-\frac{1}{2}\right)} w_{1l} \circ \phi_e^{-1}(x) = \frac{1}{2} w_{1l} \circ \phi_e^{-1}(x)$$

Graphs of quadratic wavelet basis W_0, W_1, and W_2 are shown in Fig. 2.

In this work, we will apply these two types of wavelet basis to solve the singularly perturbed problem based on the multilevel augmentation method described in the next section.

e	w_{1l}	$\Phi_e^{-1}(x)$	w_{2j}
0	w_{10}	$2x$	$w_{20}(x) = \begin{cases} \frac{1}{\sqrt{2}}(2x)(1-6x) & ; x \in \left[0, \frac{1}{4}\right) \\ \frac{1}{\sqrt{2}}(1-2x)(6x-2) & ; x \in \left[\frac{1}{4}, \frac{1}{2}\right] \end{cases}$
0	w_{11}	$2x$	$w_{21}(x) = \begin{cases} \frac{\sqrt{3}}{\sqrt{2}}(2x)(1-4x) & ; x \in \left[0, \frac{1}{4}\right) \\ \frac{\sqrt{3}}{\sqrt{2}}(1-2x)(1-4x) & ; x \in \left[\frac{1}{4}, \frac{1}{2}\right] \end{cases}$
1	w_{10}	$2x-1$	$w_{22}(x) = \begin{cases} \frac{1}{\sqrt{2}}(4-6x)(2x-1) & ; x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \frac{1}{\sqrt{2}}(2-2x)(6x-5) & ; x \in \left[\frac{3}{4}, 1\right] \end{cases}$
1	w_{11}	$2x-1$	$w_{23}(x) = \begin{cases} \frac{\sqrt{3}}{\sqrt{2}}(2x-1)(3-4x) & ; x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \frac{\sqrt{3}}{\sqrt{2}}(2-2x)(3-4x) & ; x \in \left[\frac{3}{4}, 1\right] \end{cases}$

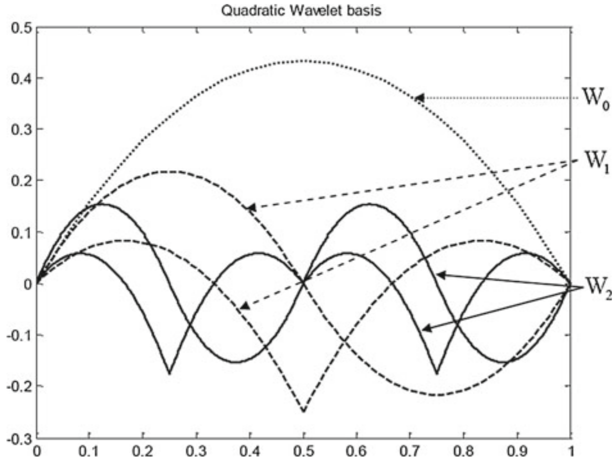

Fig. 2 Graphs of quadratic wavelet basis W_0, W_1 and W_2

3 Multilevel augmentation method

To demonstrate the applications of multilevel augmentation method for solving the singularly perturbed problem, we consider the second-order boundary value problem in the form of

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - u = 1, \quad 0 < x < 1, \tag{6}$$

where ε is perturbation parameter ($0 < \varepsilon \ll 1$). We will show the derivation of multilevel augmentation method to this simplified equation. Extended domain or higher-order derivatives involved can be considered in a similar way.

Boundary conditions are specified by

$$u(0) = 0 \text{ and } u(1) = 0. \tag{7}$$

Approximate solution is assumed to be

$$u\left(x\right)=\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)}w_{ij}\left(x\right)c_{ij},\tag{8}$$

where c_{ij} are the coefficients to be approximated, $w_{ij}\left(x\right)$ is the wavelet basis, M is the number of level in the multilevel wavelet.

Setting $\{\mathrm{w}\}=\left\{w_{ij}\left(x\right)\right\}$ and $\{c\}=\left\{c_{ij}\right\}$, for $i=1,2,3,\ldots,M$, i is the ith level, $j=0,1,\ldots,\dim \mathrm{W}_i-1$. Equation (6) can be written in a matrix form as

$$\left\{[\varphi_{\mathrm{B}r}]-[\varphi_{\mathrm{C}r}]+[\varphi_{\mathrm{A}r}]\right\}\left\{c_{ij}\right\}=-\left\{\varphi_{\mathrm{E}r}\right\},\tag{9}$$

where

$$[\varphi_{\mathrm{A}r}]=\int_0^1\{\mathrm{w}\}\{\mathrm{w}\}^T\,dx\tag{10}$$

$$[\varphi_{\mathrm{B}r}]=\varepsilon\int_0^1\left\{\frac{d\mathrm{w}}{dx}\right\}\left\{\frac{d\mathrm{w}}{dx}\right\}^T\,dx\tag{11}$$

$$[\varphi_{\mathrm{C}r}]=\int_0^1\{\mathrm{w}\}\left\{\frac{d\mathrm{w}}{dx}\right\}^T\,dx\tag{12}$$

$$\{\varphi_{\mathrm{E}r}\}=\int_0^1\{\mathrm{w}\}\,dx\tag{13}$$

Here $[\varphi_{\mathrm{A}r}]$, $[\varphi_{\mathrm{B}r}]$, $[\varphi_{\mathrm{C}r}]$, and $\{\varphi_{\mathrm{E}r}\}$ are coefficient matrices. Subscript r denotes to linear wavelet (if r is l) or the quadratic wavelet (if r is q).

Equation (9) can be written in a linear system as

$$[\mathrm{P}]\left\{c_{ij}\right\}=\{\mathrm{S}\},\tag{14}$$

where $[\mathrm{P}]=\left\{[\varphi_{\mathrm{B}r}]-[\varphi_{\mathrm{C}r}]+[\varphi_{\mathrm{A}r}]\right\}$, $\{\mathrm{S}\}=-\{\varphi_{\mathrm{E}r}\}$. We have to solve this linear system to find $\left\{c_{ij}\right\}$ for $i=1,2,3,\ldots$, and $j=0,1,\ldots,\dim \mathrm{W}_i-1$. Hence, we can obtain approximate solution.

Traditionally, the system of linear equation (14) can be solved iteratively by any standard schemes. Size of linear system depends on the number of unknowns resulting from the number of basis level. Since, we have applied multilevel approximation, it is easily to take the advantage of augmentation concept to reduce the number of unknowns in the linear system at each level. This is the concept of multilevel augmentation method. Summarized details are given as follows.

Step 1:

Solve $\left\{c_{ij}\right\}$ by multilevel method of level M from equation (14).

Step 2:
Set $\left\{c_{ij}^0\right\} = \{c_{ij}\}$ for level M.

Step 3:
Solve $\left\{c_{nj}^1\right\}$ for level $M+1$ by the multilevel augmentation method from

$$[\mathrm{P}]_{(r\times r)}\left\{c_{nj}^1\right\}_{(r\times 1)} = \{\mathrm{S}\}_{(r\times 1)}\,. \tag{15}$$

From Equation (15), splitting matrices $[\mathrm{P}]_{r\times r}$, $\left\{c_{nj}^1\right\}_{r\times 1}$ and $\{\mathrm{S}\}_{r\times 1}$ as

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix}_{(r\times r)} \left\{ \begin{matrix} c_{ij}^1 \\ c_{(M+1)j}^1 \end{matrix} \right\}_{(r\times 1)} = \left\{ \begin{matrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{matrix} \right\}_{(r\times 1)}, \tag{16}$$

where

$$\begin{aligned} &\mathrm{A}=[\mathrm{A}]_{e\times e}\,, \quad \mathrm{B}=[\mathrm{B}]_{e\times(r-e)}\,, \\ &\mathrm{C}=[\mathrm{C}]_{(r-e)\times e}\,, \quad \mathrm{D}=[\mathrm{D}]_{(r-e)\times(r-e)}\,, \\ &e=\dim \mathrm{X}_M, \quad r=\dim \mathrm{X}_{M+1}. \end{aligned}$$

Equation (16) can be rewritten as

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{D} \end{bmatrix}_{r\times r} \left\{ \begin{matrix} c_{ij}^1 \\ c_{(M+1)j}^1 \end{matrix} \right\}_{r\times 1} + \begin{bmatrix} 0 & 0 \\ \mathrm{C} & 0 \end{bmatrix}_{r\times r} \left\{ \begin{matrix} c_{ij}^1 \\ 0 \end{matrix} \right\}_{r\times 1} = \left\{ \begin{matrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{matrix} \right\}_{r\times 1},$$

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{D} \end{bmatrix}_{r\times r} \left\{ \begin{matrix} c_{ij}^1 \\ c_{(M+1)j}^1 \end{matrix} \right\}_{r\times 1} + \left\{ \begin{matrix} 0 \\ \mathrm{C}\left(c_{ij}^1\right) \end{matrix} \right\}_{r\times 1} + \left\{ \begin{matrix} 0 \\ \mathrm{C}\left(c_{ij}^0\right) \end{matrix} \right\}_{r\times 1} - \left\{ \begin{matrix} 0 \\ \mathrm{C}\left(c_{ij}^0\right) \end{matrix} \right\}_{r\times 1}$$
$$= \left\{ \begin{matrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{matrix} \right\}_{r\times 1},$$

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{D} \end{bmatrix}_{r\times r} \left\{ \begin{matrix} c_{ij}^1 \\ c_{(M+1)j}^1 \end{matrix} \right\}_{r\times 1} + \left\{ \begin{matrix} 0 \\ \mathrm{C}\left(c_{ij}^0\right) \end{matrix} \right\}_{r\times 1} = \left\{ \begin{matrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{matrix} \right\}_{r\times 1} - \left\{ \begin{matrix} 0 \\ \mathrm{C}\left(c_{ij}^1 - c_{ij}^0\right) \end{matrix} \right\}_{r\times 1},$$

where $\left\{ \begin{matrix} 0 \\ \mathrm{C}\left(c_{ij}^1 - c_{ij}^0\right) \end{matrix} \right\}_{r\times 1}$ is the error vector in the multilevel augmentation method. This error term converges to zero when the number of the basis level is very large, see proof in Chen et al. [5].

Next, we approximate the augmented system by

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{D} \end{bmatrix}_{r\times r} \left\{ \begin{matrix} c_{ij}^1 \\ c_{(M+1)j}^1 \end{matrix} \right\}_{r\times 1} = \left\{ \begin{matrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{matrix} \right\}_{r\times 1} - \left\{ \begin{matrix} 0 \\ \mathrm{C}\left(c_{ij}^0\right) \end{matrix} \right\}_{r\times 1}. \tag{17}$$

Thus, we can calculate $\left\{c_{ij}^1\right\}_{e\times 1}$ and $\left\{c_{(M+1)j}^1\right\}_{(r-e)\times 1}$ from Eq. (17) by

$$[\mathrm{D}]_{(r-e)\times(r-e)}\left\{c^{1}_{(M+1)j}\right\}_{(r-e)\times 1}=\left\{\mathrm{S}_{(M+1)j}\right\}_{(r-e)\times 1}-[\mathrm{C}]_{(r-e)\times e}\left\{c^{0}_{ij}\right\}_{e\times 1}, \tag{18}$$

$$[\mathrm{A}]_{e\times e}\left\{c^{1}_{ij}\right\}_{e\times 1}=\left\{\mathrm{S}_{ij}\right\}_{e\times 1}-[\mathrm{B}]_{e\times(r-e)}\left\{c^{1}_{(M+1)j}\right\}_{(r-e)\times 1}. \tag{19}$$

We solve system (18) directly to find $\left\{c^{1}_{(M+1)j}\right\}_{(r-e)\times 1}$, and then substituting these values into the RHS of Eq. (19) to find $\left\{c^{1}_{ij}\right\}_{e\times 1}$. This completes the overall steps in the multilevel augmentation method for level $M+1$. It shows that computational cost at level $M+1$ is reduced when comparing with standard multilevel method by the use of augmented systems (18) and (19).

For example, consider the multilevel augmentation method of level 3 with linear wavelet basis.

Step 1:

Solve $\left\{c_{ij}\right\}$ in the multilevel method of level 2 ($i=1,2$ for linear wavelet bases and $j=0,1,\ldots,\dim \mathrm{W}_i-1$) from equation

$$[\mathrm{P}]_{(3\times 3)}\left\{c_{ij}\right\}_{(3\times 1)}=\{\mathrm{S}\}_{(3\times 1)},$$

Step 2:

Set $\left\{c^{0}_{ij}\right\}=\left\{c_{ij}\right\}$ for level 2,

$$\left\{\begin{matrix} c^{0}_{10}\\ c^{0}_{20}\\ c^{0}_{21}\end{matrix}\right\}=\left\{\begin{matrix} c_{10}\\ c_{20}\\ c_{21}\end{matrix}\right\}.$$

Step 3:

Solve $\left\{c^{1}_{nj}\right\}$ for level 3 by the multilevel augmentation method,

$$[\mathrm{P}]_{(7\times 7)}\left\{c^{1}_{nj}\right\}_{(7\times 1)}=\{\mathrm{S}\}_{(7\times 1)}.$$

Splitting [P] to matrices [A] , [B] , [C] and [D],

$$[\mathrm{P}]_{(7\times 7)}\left\{c^{1}_{nj}\right\}_{(7\times 1)}=\{\mathrm{S}\}_{(7\times 1)},$$

$$\begin{bmatrix} \mathrm{A} & \mathrm{B}\\ \mathrm{C} & \mathrm{D}\end{bmatrix}_{(7\times 7)}\left\{c^{1}_{nj}\right\}_{(7\times 1)}=\{\mathrm{S}\}_{(7\times 1)},$$

Splitting matrices $\left\{c^{1}_{nj}\right\}_{(7\times 1)}$ and $\{\mathrm{S}\}_{(7\times 1)}$ by setting

$$\left\{c^{1}_{nj}\right\}_{(7\times 1)}=\left\{\begin{matrix} c^{1}_{ij}\\ c^{1}_{(M+1)j}\end{matrix}\right\}_{(7\times 1)},$$

$$\{\mathrm{S}\}_{(7\times 1)}=\left\{\begin{matrix} \mathrm{S}_{ij}\\ \mathrm{S}_{(M+1)j}\end{matrix}\right\}_{(7\times 1)}.$$

Solve $\left\{c_{nj}^1\right\}_{7\times1}$ for level 3 from equation

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{D} \end{bmatrix}_{(7\times7)} \left\{ \begin{array}{c} c_{ij}^1 \\ c_{(M+1)j}^1 \end{array} \right\}_{(7\times1)} = \left\{ \begin{array}{c} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{array} \right\}_{(7\times1)} - \left\{ \begin{array}{c} 0 \\ \mathrm{C}\left(c_{ij}^0\right) \end{array} \right\}_{(7\times1)},$$

Find $\left\{c_{(M+1)j}^1\right\}_{4\times1}$ of level 3 from equation

$$[\mathrm{D}]_{(4\times4)} \left\{c_{(M+1)j}^1\right\}_{(4\times1)} = \left\{\mathrm{S}_{(M+1)j}\right\}_{(4\times1)} - [\mathrm{C}]_{(4\times3)} \left\{c_{ij}^0\right\}_{(3\times1)},$$

This system can be solved easily since the matrix coefficient is diagonal for linear wavelet and becomes tri-diagonal for quadratic wavelet.

Finally, solve $\left\{c_{ij}^1\right\}_{3\times1}$ of level 3 from equation

$$[\mathrm{A}]_{(3\times3)} \left\{c_{ij}^1\right\}_{(3\times1)} = \left\{\mathrm{S}_{ij}\right\}_{(3\times1)} - [\mathrm{B}]_{(3\times4)} \left\{c_{(M+1)j}^1\right\}_{(4\times1)}.$$

Thus, we can find all coefficients in the solution expansion for level 3. Hence, we use these values to find coefficients in the next level until finishing process.

4 Numerical results

Singularly perturbed problem is represented by

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - u = 1, \quad 0 < x < 1.$$

Here we set $\boldsymbol{\varepsilon} = 0.1$, subject to the boundary conditions,

$$u(0) = 0 \quad \text{and} \quad u(1) = 0.$$

The exact solution is

$$u(x) = \frac{(e^{m_2} - 1)\, e^{m_1 x} + (1 - e^{m_1})\, e^{m_2 x}}{e^{m_2} - e^{m_1}} - 1,$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}, \quad m_2 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}.$$

To study the accuracy of present scheme, we use the RMS error defined by

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{N} (u_i - u_{Exact})^2}{N}}, \tag{20}$$

and L^2 norms defined by

$$\|u_i - u_{Exact}\|_{L^2} = \left(\sum_{i=1}^{N} (u_i - u_{Exact})^2 \right)^{\frac{1}{2}}. \tag{21}$$

where $u_i = u(x_i)$, x_i are knots in wavelet basis.

We apply the linear and quadratic wavelet bases in our approximations. Numerical results are shown in Tables 1, 2, 3 and 4. The multilevel augmentation method is performed at the initial level 3 (W_3) for both linear and quadratic bases. The systems of linear equations are solved iteratively by the Gauss–Seidel method with $TOL = 10^{-12}$.

The RMS errors by the multilevel augmentation method are almost the same as those errors obtained by using the multilevel method. Rate of convergence is approximately order two as expected for the case of linear wavelet while the rate of convergences for the quadratic wavelet is approximately order four. The comparison between numerical result and exact solution is shown in Fig. 3. They are in good agreement. In this case, numerical results are obtained from applying linear basis with level 9.

Table 1 RMS error by linear wavelet

Level	Multilevel augmentation method	Multilevel method
W_4	3.683109e−03	3.475348e−03
W_5	8.281858e−04	8.337140e−04
W_6	2.042934e−04	2.053123e−04
W_7	5.096677e−05	5.103239e−05
W_8	1.272315e−05	1.272720e−05
W_9	3.178067e−06	3.178319e−06
W_{10}	7.941528e−07	7.941685e−07
W_{11}	1.984912e−07	1.984922e−07
W_{12}	4.961684e−08	–
Convergence rate	2.0	2.0

Table 2 L^2 norms by linear wavelet

Level	Multilevel augmentation method	Multilevel method
W_4	1.426462e−02	1.345996e−02
W_5	4.611143e−03	4.641923e−03
W_6	1.621528e−03	1.629616e−03
W_7	5.743663e−04	5.751059e−04
W_8	2.031724e−04	2.032371e−04
W_9	7.184120e−05	7.184689e−05
W_{10}	2.540047e−05	2.540098e−05
W_{11}	8.980494e−06	8.980538e−06
W_{12}	3.175090e−06	–
Convergence rate	1.5	1.5

Table 3 RMS error by quadratic wavelet

Level	Multilevel augmentation method	Multilevel method
W_4	6.335132e−05	4.037317e−05
W_5	3.645426e−06	2.580418e−06
W_6	1.785070e−07	1.618866e−07
W_7	1.029555e−08	1.011768e−08
Convergence rate	4.0	4.0

Table 4 L^2 norms by quadratic wavelet

Level	Multilevel augmentation method	Multilevel method
W_4	5.652035e−04	2.247883e−04
W_5	2.893467e−05	2.048143e−05
W_6	2.011672e−06	1.824370e−06
W_7	1.644069e−07	1.615664e−07
Convergence rate	3.5	3.5

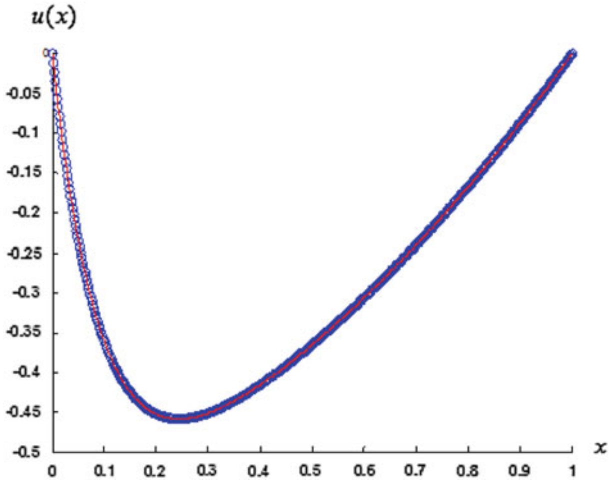

Fig. 3 Numerical results by linear wavelet level 9 (*circle*) and exact solution (*line*)

Numerical error represented by L^2 norms for the multilevel augmentation method is almost the same as those obtained from standard multilevel method. Rate of convergence is approximately 1.5 as expected for the linear wavelet. Rate of convergence for quadratic wavelet is shown. It is approximately in order 3.5.

Numerical results show that the proposed multilevel augmentation method is fast and accurate for solving the problem. It has great advantage in the case of large unknowns, but the multilevel augmentation method should be applied at high-initial level to reduce error collection for each upper level. The full system of multilevel augmentation method is smaller than the standard multilevel method at the same level, resulting to much smaller memory to store matrix coefficients. This explanation is shown in Table 1 for the linear basis. We cannot solve the linear system by the Gauss–Seidel method in Matlab for the system of level 12. It has no enough memory in our computer, Intel Core i5-2410M CPU and 4.00 GB memory. But the multilevel augmentation method can provide numerical results.

5 Conclusions

Multilevel augmentation method using wavelet bases to solve numerically the singularly problem is presented. This method is fast and accurate for solving linear differential equations. It has great advantage for solving the case of large unknowns. It integrates the choices of basis and designs numerical solver resulting to linear system. We consider both linear and quadratic wavelet bases. Numerical results are presented to demonstrate the efficiency of this method. The RMS errors by the multilevel augmentation method are almost the same as those obtained from the standard multilevel method. Rate of convergence is approximately order two for linear basis while it is approximately order 1.5 in L^2 norm sence. Rate of convergences for quadratic wavelet is approximately in order four while it is approximately order 3.5 in L^2 norms. It is found that the multilevel augmentation method should be applied for relatively high-initial level to reduce error collection at each level in augmented system. Full system of multilevel augmentation method is smaller than the standard multilevel method at the same level, resulting to smaller memory to store matrix coefficients.

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